UPPER SEMICONTINUITY OF TRAJECTORY ATTRACTORS OF 3D HYPERVISCOSFLOW

ABDELHAFID YOUNSI

Abstract. We regularized the 3D Navier-Stokes equations by adding a high-order viscosity term. We study the upper semicontinuity, of the global attractors of the Leray-Hopf weak solutions of the regularized 3D Navier-Stokes equations, as the artificial dissipation ε goes to 0. We also consider applications of obtained results to the regularized problem by allowing the family of forcing functions to vary with ε, for $\varepsilon > 0$.

1. Introduction

In this paper, we study the robustness, or upper semicontinuity of the global attractors of the Leray-Hopf weak solutions of modified three dimensional Navier-Stokes equations. We regularized the 3D Navier-Stokes system by adding a high order artificial viscosity term to the conventional system

$$\begin{gathered}\frac{\partial u^\varepsilon}{\partial t}+\varepsilon(-\triangle)^l u^\varepsilon-\nu\triangle u^\varepsilon+(u^\varepsilon.\nabla)\,u^\varepsilon+\nabla p=f\,(x)\,,\ \text{in } \Omega\times(0,\infty)\\ \text{div } u^\varepsilon=0,\ \text{in } \Omega\times(0,\infty)\,,\ u^\varepsilon\,(x,0)=u_0^\varepsilon,\ \text{in } \Omega,\\ p(x+Le_i,t)=p(x,t),\ u^\varepsilon(x+Le_i,t)=u^\varepsilon(x,t)\ \ i=1,2\ ,3,\ t\in(0,\infty)\end{gathered}\tag{1.1}$$

where $\Omega = (0, L)^3$ with periodic boundary conditions and $(e_1, ..., e_d)$ is the natural basis of $\mathbb{R}^d$. Here $\varepsilon > 0$ is the artificial dissipation parameter, u^ε is the velocity vector field, p is the pressure, $\nu > 0$ is the kinematic viscosity of the fluid and f is a given force field. For $\varepsilon = 0$, the model is reduced to 3D Navier–Stokes system.

Mathematical model for such fluid motion has been used extensively in turbulence simulations (see e.g. [3, 4, 7, 10]). For further discussion of theoretical results concerning (1.1), see [1, 2, 5, 12, 15, 16, 20, 23].

In the work [23], the strong convergence of the solution of this problem to the solution of the conventional system as the regularization parameter goes to zero, was established for each dimension $d \leq 4$.

For the 3D Navier–Stokes system weak solutions of problem are known to exist by a basic result by J. Leray from 1934 [11], only the uniqueness of weak solutions remains as an open problem. Then the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D Navier–Stokes system.

The theory of trajectory attractors for evolution partial differential equations was developed in [14, 18], which the uniqueness theorem of solutions of the corresponding initial-value problem is not proved yet, e.g. for the 3D Navier–Stokes system (see [8, 14, 17, 18]). Such trajectory attractor is a classical global attractor

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but in the space of weak solutions defined on $[0, \infty)$, with the corresponding semi-group being simply the translation in time of such solutions. A compact set $\mathfrak{A} \Subset E$ is said to be a global attractor of a semigroup $\{S(t), t > 0\}$ acting in a Banach or Hilbert space E if $\mathfrak{A}$ is strictly invariant with respect to $\{S(t)\} : S(t)\mathfrak{A} = \mathfrak{A}\ \forall t \geq 0$ and $\mathfrak{A}$ attracts any bounded set $B \subset E : dist(S(t)B, \mathfrak{A}) \to 0\ (t \to \infty)$ (see [13], [14], [17], [18], [20]).

In this article, we study the upper semicontinuity, of the global attractors of the Leray-Hopf weak solutions of a regularized 3D Navier-Stokes equations, as the artificial dissipation ε goes to 0. While there exist other examples of such robustness in the literature of the Navier-Stokes equations, the specific emphasis on the regularized problem is new for the 3D Navier-Stokes equations and is of interest. This would bean extension of the earlier work on Ou and Sritharan for the 2D Navier-Stokes equations, see references [15] and [16]. It is now known that there is a global attractor $\mathfrak{A}_0$ for the Leray-Hopf weak solutions of the 3D Navier-Stokes equations, see Sell [17] or [18].

The main object of this paper to show that there is a global attractor, which one might denote by $\mathfrak{A}_\varepsilon$, for the regularized problem (1.1), and that the family $\{\mathfrak{A}_\varepsilon\}$ is upper semicontinuous at $\varepsilon = 0$. Moreover, we can modify the argument described above so that the final result will have broader applicability by allowing the family of forcing functions f^ε to vary with ε, for $\varepsilon > 0$.

The family of sets $\mathfrak{A}_\varepsilon$, $0 < \varepsilon \leq 1$ is robust at $\mathfrak{A}_0$, or is upper semicontinuous with respect to ε at $\varepsilon_0 = 0$, provided that, for every $\varepsilon_0 > 0$, there is a neighborhood $O(\varepsilon_0)$ of $0 \in \mathbb{R}$ and a neighborhood $N_{\varepsilon_0}(\mathfrak{A}_0)$ of $\mathfrak{A}_0$, such that $\mathfrak{A}_\varepsilon \subset N_{\varepsilon_0}(\mathfrak{A}_0)$, for every $\varepsilon \in O(\varepsilon_0)$ with $\varepsilon > 0$, see (23.13) in [18].

The paper is organized as follows. In Section 2, we present the relevant mathematical framework for the paper. In Section 3, we recall the definition of the trajectory attractor $\mathfrak{A}_0$ of the conventional 3-D Navier-Stokes equations. In Section 4, we study the regularized problem (see equation (1.1)), then we show the existence of trajectory attractor $\mathfrak{A}_\varepsilon$. In Section 5, we present the main result of this paper, that is, a theorem on the upper semicontinuity on the attractors $\mathfrak{A}_\varepsilon$. Finally, an application of our general results to the study of the robustness of the system (1.1) with a perturbed external force.

2. Preliminary

We denote by $H^m(\Omega)$, the Sobolev space of L-periodic functions endowed with the inner product

$$(u, v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)} \text{ and the norm } \|u\|_m = \sum_{|\beta| \leq m} (\left\|D^\beta u\right\|^2_{L^2(\Omega)})^{\frac{1}{2}}$$

and by $H^{-m}(\Omega)$ the dual space of $H^m(\Omega)$. We denote by $\dot{H}^m(\Omega)$ the subspace of $H^m(\Omega)$ with, zero average $\dot{H}^m(\Omega) = \{u \in H^m(\Omega) ; \int_\Omega u(x)\, dx = 0\}$.

- We introduce the following solenoidal subspaces V_s, $s \in \mathbb{R}^+$ which are important to our analysis

$$V_0(\Omega) = \{u \in \dot{L}^2(\Omega), \mathrm{div} u = 0, u.n\,|_{\Sigma_i} = -u.n\,|_{\Sigma_{i+3}}, i = 1, 2, 3\};$$

$$V_1(\Omega) = \{u \in \dot{H}^1(\Omega), \mathrm{div} u = 0, \gamma_0 u\,|_{\Sigma_i} = \gamma_0 u\,|_{\Sigma_{i+3}}, i = 1, 2, 3\}.$$

$$V_2(\Omega) = \{u \in \dot{H}^2(\Omega), \mathrm{div} u = 0, \gamma_0 u\,|_{\Sigma_i} = \gamma_0 u\,|_{\Sigma_{i+3}}, \gamma_1 u\,|_{\Sigma_i} = -\gamma_1 u\,|_{\Sigma_{i+3}}, i = 1, 2, 3\}.$$

see [20]. We refer the reader to Temam [21] for details on these spaces. Here the faces of Ω are numbered as

$$\Sigma_i = \partial\Omega \cap \{x_i = 0\} \text{ and } \Sigma_{i+3} = \partial\Omega \cap \{x_i = L\}, \ i = 1, 2, 3.$$

Here γ_0, γ_1 are the trace operators and n is the unit outward normal on $\partial\Omega$.

- The space V_0 is endowed with the inner product $(u, v)_{L^2(\Omega)}$ and norm $\|u\| = (u, u)^{1/2}_{L^2(\Omega)}$.
- V_1 is the Hilbert space with the norm $\|u\|_1 = \|u\|_{V_1}$. The norm induced by $\dot{H}^1(\Omega)$ and the norm $\|\nabla u\|_{L^2(\Omega)}$ are equivalent in V_1.
- V_2 is the Hilbert space with the norm $\|u\|_2 = \|u\|_{V_2}$. In V_2 the norm induced by $\dot{H}^2(\Omega)$ is equivalent to the norm $\|\triangle u\|_{L^2(\Omega)}$.

V_s' denote the dual space of V_s. We present the topology to be used for generating the neighborhood of robustness. Let F any vector space. A metric $d(f, g)$ on F is said to be invariant if one has

$$d(f, g) = d(f - g, 0) \text{ for all } f, g \in F.$$

A Fréchet space is a complete topological vector space whose topology is induced by a translation invariant metric $d(f, g)$. Given a Banach space X, with norm $\|.\|_X$ and $1 \leq p < \infty$, we denote by $L^p_{loc}[0, \infty; X)$ the Fréchet space of mesurable functions $f : [0, \infty) \to X$ that are p-integrable over $[0, T]$, for each $0 < T < \infty$, endow with the metric

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \min(\|f - g\|_{L^p(0,\ n;\ X)}, 1).$$

We denote by $L^p_{loc}(0, \infty; X)$ the Fréchet space of mesurable functions $f : (0, \infty) \to X$ that are p-integrable over $[t_0, T]$, for each $0 < t_0 \leq T < \infty$ endow with the metric

$$d(f, g) = \sum_{n=2}^{\infty} 2^{-n} \min(\|f - g\|_{L^p(\frac{1}{n},\ n;\ X)}, 1).$$

Similarly for $p = \infty$, we will let $L^\infty_{loc}(0, \infty; X)$ denote the collection of all functions $f : (0, \infty) \to X$ with the property that, for all τ and T with $0 < T < \infty$, one has $\operatorname{ess} \sup_{0<s<T} \|f\|_X < \infty$. We denote by $C[0, \infty; X)$ the space of strongly continuous functions from $[0, \infty)$ to X, endowed with the topologie of the uniform convergence over compact sets and by $C_w[0, \infty; X)$ the space of weakly continuous functions from $[0, \infty)$ to X. We denote by $L^\infty C = L^\infty(\mathbb{R}, X) \cap C(\mathbb{R}, X)$ the Fréchet space $L^\infty C$ endow with the $L^\infty_{loc}-$topology, wich is the topology of uniform convergence on bounded sets.

Let E be a complete metric space with metric d. We write B_r for the open ball centre $0 \in E$ and radius r. The following quantity is called the Hausdorff (non-symmetric) semidistance from a set X to a set Y in a Banach space E

$$dist_E(X, Y) = \sup_{x\in X} \inf_{y\in Y} \|.\|_E.$$

Let M be a subset of E and let $\mathbb{R}^+ = [0, \infty)$. A mapping $\sigma = \sigma(u, t)$, where $\sigma : M \times [0, \infty) \to M$ is said to be a semiflow on M provided the following hold

1) $\sigma(w,0) = w$, for all $w \in M$.
2) The semigroup property holds, i. e,

$$\sigma((w,s),t) = \sigma(w,s+t) \text{ for all } w \in M \text{ and } s,\, t \in \mathbb{R}^+.$$

3) The mapping $\sigma : M \times (0,\infty) \to M$ is continuous.

If in addition the mapping $\sigma : M \times [0,\infty) \to M$ is continuous we will say that the semiflow is continuous at $t = 0$. Here we use $t > 0$ in order that the Robustness Theorem 23.14 in [18] is valid, see Sell [18] and Hale [8]. For any $u \in M$ the positive trajectory through u is defined as the set $\gamma^+(u) = \{\sigma(t)u,\ t \geq 0\}$. For any set $B \subset M$ we define the positive hull $\mathcal{H}^+(B)$ and the omega limit set $\omega(B)$ as follows

$$\mathcal{H}^+(B) = Cl_M \gamma^+(B) \text{ and } \omega(B) = \cap_{\tau \geq 0} \mathcal{H}^+(\sigma(\tau)B).$$

If $\mathcal{A} \subset E$ and $\varepsilon > 0$ we write

$$N_\varepsilon(\mathcal{A}) = \{z \in E,\ \inf_{a \in \mathcal{A}} d(z,a) < \varepsilon\}.$$

for the open $\varepsilon-$neighbourhood of $\mathcal{A}$.

We denote by A the Stokes operator $Au = -\triangle u$ for $u \in D(A)$.We recall that the operator A is a closed positive self-adjoint unbounded operator, with $D(A) = \{u \in V_0,\, Au \in V_0\}$. We have in fact, $D(A) = \dot{H}^2(\Omega) \cap V_0 = V_2$. The spectral theory of A allows us to define the powers A^l of A for $l \geq 1$, A^l is an unbounded self-adjoint operator in V_0 with a domain $D(A^l)$ dense in $V_2 \subset V_0$. We set here

$$A^l u = (-\triangle)^l u \text{ for } u \in D\left(A^l\right) = V_{2l} \cap V_0.$$

The space $D\left(A^l\right)$ is endowed with the scalar product and the norm

$$(u,v)_{D(A^l)} = \left(A^l u, A^l v\right), \|u\|_{D(A^l)} = \{(u,u)_{D(A^l)}\}^{\frac{1}{2}}.$$

Now define the trilinear form $b(.,.,.)$ associated with the inertia terms

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx$$

The continuity property of the trilinear form enables us to define (using Riesz representation theorem) a bilinear continuous operator $B(u,v)$; $V_2 \times V_2 \to V_2'$ will be defined by

$$\langle B(u,v), w\rangle = b(u,v,w),\ \forall w \in V_2.$$

Recall that for u satisfying $\nabla.u = 0$ we have

$$b(u,u,u) = 0 \text{ and } b(u,v,w) = -b(u,w,v). \tag{2.1}$$

Hereafter, $c_i \in \mathbb{N}$,will denote a dimensionless scale invariant positive constant which might depend on the shape of the domain. The trilinear form $b(.,.,.)$ is continuous on $\dot{H}^{m_1}(\Omega) \times \dot{H}^{m_2+1}(\Omega) \times \dot{H}^{m_3}(\Omega)$, $m_i \geq 0$

$$|b(u,v,w)| \leq c_0 \|u\|_{m_1} \|v\|_{m_2+1} \|w\|_{m_3} \quad ,\ m_3 + m_2 + m_1 \geq \frac{3}{2} \tag{2.2}$$

see [21]. Similarly, the trilinear form $b(u,v,w)$ satisfies the well-known inequalities (see, for instance, [20, Lemma 61.1] and [7, 21])

$$|b(u,v,u)| \leq c_1 \|u\|^{\frac{1}{2}} \|u\|_1^{\frac{3}{2}} \|v\|_1 \text{ for all } u,v \in V. \tag{2.3}$$

Similarly, we define $\hat{B}(u,v) \in V_1'$ by

$$\left\langle \hat{B}(u,v), w \right\rangle_{V_1' \times V_1} = b(u,v,w), \ \forall w \in V_1.$$

We recall some inequalities that we will be using in what follows.

Agmon inequality (see, e.g., [7])

$$\|u\|_\infty \le c_3 \|u\|_1^{\frac{1}{2}} \|Au\|^{\frac{1}{2}} \text{ for all } u \in V_2. \tag{2.4}$$

Young's inequality

$$ab \le \frac{\epsilon}{p} a^p + \frac{1}{q\epsilon^{\frac{q}{p}}} b^q, a, b, \epsilon > 0, p > 1, q = \frac{p}{p-1}. \tag{2.5}$$

Poincaré's inequality

$$\lambda_1 \|u\|^2 \le \|u\|_1^2 \text{ for all } u \in V_0, \tag{2.6}$$

where λ_1 is the smallest eigenvalue of the Stokes operator A.

3. Navier-Stokes equations

The conventional Navier-Stokes system can be written in the evolution form

$$\begin{gathered} \frac{\partial u}{\partial t} + \nu A u + \hat{B}(u,u) = f, \ t > 0, \\ u_0(x) = u_0. \end{gathered} \tag{3.1}$$

Let $f \in L^\infty(0,\infty;V_0)$ be given. We will say that a function u is a weak solution of the 3D Navier-Stokes of Class LH (Leray–Hopf) on $[0,\infty)$ provided that $u(x,0) = u_0(x) \in V_0$, and the following properties hold

1) $u \in L^\infty(0,\infty;V_0) \cap L^2_{loc}[0,\infty;V_1)$.

2) $\dfrac{du}{dt} \in [L^{\frac{4}{3}}_{loc} 0,\infty;V_1^{'})$.

Taking the inner product of (3.1) with u, and using (2.5) we have

$$\frac{d}{dt}\|u(t)\|^2 + 2\nu \|\nabla u\|^2 = 2\langle f, u \rangle. \tag{3.2}$$

by application of Young's inequality and the Poincaré's Lemma, yields

$$\frac{d}{dt}\|u(t)\|^2 + \nu \|\nabla u\|^2 \le \frac{\|f\|^2}{\nu\lambda_1}, \tag{3.3}$$

using the Poincaré Lemma and Gronwall's inequality, to get

$$\|u(t)\|^2 \le e^{-\nu\lambda_1(t-t_0)}\|u(t_0)\|^2 + \frac{1}{\nu^2\lambda_1^2}\|f\|^2 \left(1 - e^{-\nu\lambda_1(t-t_0)}\right), \text{with } 0 < t_0 < t,$$

3) which implies that

$$\|u(t)\|^2 \le e^{-\nu\lambda_1(t-t_0)}\|u(t_0)\|^2 + \frac{1}{\nu^2\lambda_1^2}\|f\|^2. \tag{3.4}$$

Integrating (3.2) over $[t_0, t]$ we find that

$$\|u(t)\|^2 + 2\nu \int_{t_0}^t \|A^{\frac{1}{2}} u(s)\|^2 ds \le \|u(t_0)\|^2 + 2\int_{t_0}^t \langle f(s), u(s) \rangle ds. \tag{3.5}$$

4) The function u satisfies the following equality

$$\langle u(t)-u(t_0),v\rangle+\nu\int_{t_0}^{t}\left\langle A^{\frac{1}{2}}u(s),A^{\frac{1}{2}}v\right\rangle ds+\int_{t_0}^{t}\left\langle \hat{B}(u(s),u(s)),v\right\rangle ds=\int_{t_0}^{t}\langle f,v\rangle ds, \tag{3.6}$$

for all $v\in V_1$ and for all $t\geq t_0\geq 0$.

The proof of the following theorem is given in [12, 13, 21].

Theorem 3.1. *Let $f\in V_1'$ and $u_0\in V_0$ be given. Then for every $T>0$, there exists a weak solution $u(t)$ of (3.1) from the space $L^2(0,T;V_1)\cap L^\infty(0,T;V_0)$, such that $u(x,0)=u_0$ and $u(t)$ satisfies the energy equality (3.6).*

Moreover (see [21]), $u(.)$ is weakly continuous from $[0,T]$ into V_0, the function $u\in C_w([0,T];\ V_0)$ and consequently $u(x,0)=u_0(x)\in V_0$. Let W is the set of all Leray–Hopf weak solutions $u(.)$ of equation (3.1) in the space $L^\infty(0,\infty;V_0)\cap L^2_{loc}[0,\infty;V_1)$ that satisfy the following properties

- $\dfrac{du}{dt}\in L^{\frac{4}{3}}_{loc}(0,\infty;V_1^{'})$;
- for almost all t and t_0, with $t>t_0>0$, inequalities (3.5 ,3.6) are valid.

Let X^0 denote the Fréchet space used to define the Leray-Hopf weak solutions. Thus

$$\varphi\in X^0=L^\infty(0,\infty;V_0)\cap L^2_{loc}[0,\infty;V_1),$$

where $\varphi\in C_w[0,\infty;V_0)$ and we let $\mathfrak{F}^0$ denote a compact, translation invariant set of forcing functions f in

$$L^\infty C=L^\infty\left(\mathbb{R},L^2(\Omega)\right)\cap C\left(\mathbb{R},L^2(\Omega)\right)$$

where the topology on the Fréchet space $L^\infty C$ is the topology of uniform convergence on bounded sets in $\mathbb{R}$.

Then, we use the Leray-Hopf solutions of the 3D Navier-Stokes equations with $\varepsilon=0$ to generate a semiflow π^0 on $\mathfrak{F}^0\times X^0$, where

$$\pi^0(\tau)(f,\varphi)=\left(f_\tau,S^0(f,\tau)\varphi\right)\ \text{ for }\tau\geq 0,$$

$f_\tau(t)=f(\tau+t)$ and $u(t)=S^0(f,t)\varphi$ is the Leray-Hopf solution of the 3D Navier-Stokes equations that satisfies $u(0)=S^0(f,0)\varphi=\varphi(0)$. By using the theory of generalized weak solutions, as in Sell [17] or [18] , we note that π^0 has a global attractor $\mathfrak{A}_0\subset\mathfrak{F}^0\times X^0$ see Theorem 65.12 in [18].

4. The regularized Navier-Stokes system

Using the operators defined in the previous section, we can write the modified system (1.1) in the evolution form

$$\begin{aligned}\partial_t u^\varepsilon+\varepsilon A^l u^\varepsilon+\nu A u^\varepsilon+B(u^\varepsilon,u^\varepsilon)=f(x),\quad&\text{in}\quad\Omega\times(0,\infty)\\ u^\varepsilon(x)=u^\varepsilon,\quad&\text{in}\quad\Omega.\end{aligned}\tag{4.1}$$

For $\varepsilon>0$, we let π^ε denote the semiflow on $\mathfrak{F}^0\times X^0$ generated by the weak solutions of regularized 3D Navier-Stokes equations of (4.1). Thus

$$\pi^\varepsilon(\tau)=\left(f_\tau,S^\varepsilon(f,\tau)\varphi\right),\tag{4.2}$$

where $u_0^\varepsilon=\varphi$ and

$$u^\varepsilon(t)=S^\varepsilon(f,t)\varphi=S^\varepsilon(f,t)u_0^\varepsilon\tag{4.3}$$

is the weak solution of (4.1) that satisfies $u^\varepsilon(0)=\varphi(0)=u_0^\varepsilon(0)$.

The existence and uniqueness results for initial value problem (1.1) can be found in [12, Remark 6.11].
The following theorem collects the main result in this work

Theorem 4.1. *For $l \geq \frac{5}{4}$, for $\varepsilon > 0$ fixed, $f \in L^2(0,T;V_0')$ and $u_0^\varepsilon \in V_0$ be given. There exists a unique weak solution of (4.1) which satisfies*

$$u^\varepsilon \in L^2(0,T;V_l) \cap L^\infty(0,T;V_0), \forall T > 0.$$

Then, $u^\varepsilon \in L^\infty(0,T;V_0) \cap L^2[0,T;V_1)$ and $u^\varepsilon \in C_w([0,T];V_0), \forall T > 0.$

We recall Lemma 3.7. [23].

Lemma 4.2. *u^ε is almost everywhere equal to a continuous function from $[0,T]$ to the space V_0.*

and the following theorem

Theorem 4.3. *For $l \geq \frac{3}{2}$, the weak solution u^ε of the modified Navier-Stokes equations (4.1) given by Theorem 4.1 converges strongly in $L^2(0,T;V_0)$ as $\varepsilon \to 0$ to u a weak solution of the Navier-Stokes equations.*

The above theorem is established directly by using of a general result [23, Theorem 3.9.].

Now, we show that the semigroup $S^\varepsilon(t)$ has an absorbing ball in V_0 and an absorbing ball in V_1. Then we show that $S^\varepsilon(t)$ admits a compact attractor in V_0 for each $\varepsilon \geq 0$.

We take the inner product of (4.1) with u_ε, we obtain the energy equality

$$\frac{d}{dt}\|u_\varepsilon\|^2 + 2\varepsilon\left(A^l u^\varepsilon, u^\varepsilon\right) + 2\nu\|\nabla u_\varepsilon\|^2 = 2(f, u_\varepsilon).$$

Here we have used the fact that $b(u_\varepsilon, u_\varepsilon, u_\varepsilon) = 0$. By applying Young's inequality and the Poincaré Lemma, we get

$$\frac{d}{dt}\|u_\varepsilon\|^2 + 2\varepsilon\|A^{\frac{l}{2}}u^\varepsilon\|^2 + \nu\|\nabla u_\varepsilon\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1}, \tag{4.4}$$

we drop the term $2\varepsilon\|A^{\frac{l}{2}}u^\varepsilon\|^2$, we obtain

$$\frac{d}{dt}\|u_\varepsilon\|^2 + \nu\lambda_1\|u_\varepsilon\|^2 \leq \frac{\|f\|^2}{\nu\lambda_1},$$

by integrating the above inequality from 0 to t, we get

$$\|u_\varepsilon(t)\|^2 \leq \|u_{\varepsilon 0}\|^2 e^{-\nu\lambda_1 t} + \rho_0^2\left(1 - e^{-\nu\lambda_1 t}\right),\ t > 0, \tag{4.5}$$

where $\rho_0 = \dfrac{1}{\nu\lambda_1}\|f\|$. Hence for any ball $B_{R_0} = \{u_{\varepsilon 0} \in V_0;\ \|u_{\varepsilon 0}\| \leq R_0\}$ there is a ball $B(0,\delta_0)$ in V_0 centered at origin with radius $\delta_0 > \rho_0$ $(R_0 > \delta_0)$ such that

$$S^\varepsilon(t)B_{R_0} \subset B_{r_0} \text{ for } t \geq t_0(B_{R_0}) = \frac{1}{\nu\lambda_1}\log\frac{R_0^2 - \rho_0^2}{\delta_0^2 - \rho_0^2}. \tag{4.6}$$

The ball B_{δ_0} is said to be absorbing and invariant under the action of $S^\varepsilon(t)$.

Taking the limit in (4.5) we get,

$$\limsup_{t\to\infty}\|u_\varepsilon(t)\| \leq \rho_0. \tag{4.7}$$

We integrate (4.4) from t to $t+r$, we obtain for $u_{\varepsilon 0}\in B_{R_0}$

$$\int_t^{t+r}\|u_\varepsilon\|_1^2\,ds\leq\frac{1}{\nu}(\frac{r\,\|f\|^2}{\nu\lambda_1}+\|u_\varepsilon(t)\|^2),\ \forall r>0,\ \forall t\geq t_0(B_{R_0}). \tag{4.8}$$

With the use of (4.7) we conclude that

$$\limsup_{t\to\infty}\int_t^{t+r}\|u_\varepsilon\|_1^2\,ds\leq\frac{r}{\nu^2\lambda_1}\|f\|^2+\frac{\|f\|^2}{\nu^3\lambda_1^2}, \tag{4.9}$$

from which we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\|u_\varepsilon\|_1^2\,ds\leq\frac{\|f\|^2}{\nu^2\lambda_1}, \tag{4.10}$$

this verifies that the left-hand side is finite.

To show that the semigroup $S^\varepsilon(t)$ has an absorbing set in V_1, we consider the strong solutions and take the inner product of (4.1) with Au_ε, we obtain

$$\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}u_\varepsilon\|^2+\varepsilon\left(A^lu^\varepsilon,Au^\varepsilon\right)+\nu\|Au_\varepsilon\|^2=-b(u_\varepsilon,u_\varepsilon,Au_\varepsilon)+(f,Au_\varepsilon). \tag{4.11}$$

By applying Young's inequality, we get

$$\begin{aligned}(f,Au_\varepsilon)&\leq\|f\|\,\|Au_\varepsilon\|\\&\leq\frac{\nu}{4}\|Au_\varepsilon\|^2+\frac{1}{\nu}\|f\|^2.\end{aligned}$$

By using the Agmon's inequality (2.4) and Young's inequality we can estimate the last term in the left-hand side of (4.11) as follows

$$\begin{aligned}|b(u_\varepsilon,u_\varepsilon,Au_\varepsilon)|&\leq\|u_\varepsilon\|_\infty\|u_\varepsilon\|_1\|Au_\varepsilon\|\\&\leq c_4\|u_\varepsilon\|_1^{\frac{3}{2}}\|Au_\varepsilon\|^{\frac{3}{2}}\\&\leq\frac{\nu}{4}\|Au_\varepsilon\|^2+c_4\|u_\varepsilon\|_1^6.\end{aligned}$$

Hence we obtain from (4.11)

$$\frac{d}{dt}\|u_\varepsilon\|_1^2+2\varepsilon\|A^{\frac{l+1}{2}}u_\varepsilon\|^2+\nu\|Au_\varepsilon\|^2\leq\frac{2}{\nu}\|f\|^2+2c_4\|u_\varepsilon\|_1^6.$$

Dropping the positive terms associated with ε we have

$$\frac{d}{dt}\|u_\varepsilon\|_1^2+\nu\|A_1u_\varepsilon\|^2\leq\frac{2\|f\|^2}{\nu}+2c_4\|u_\varepsilon\|_1^6 \tag{4.12}$$

we apply the uniform Gronwall Lemma to (4.12) with

$$g=2c_4\|u_\varepsilon\|_1^4,\ h=\frac{2\|f\|^2}{\nu},\ y=\|u_\varepsilon\|_1^2.$$

For $n=3$, $m=l\geq\frac{3}{2}$ and $\theta=\frac{1}{2}$, in [12, Formula (6.167)], we get $q_\theta=6$ wich means $u_\varepsilon\in L^6(0,T;V_1)$ then $u_\varepsilon\in L^4(0,T;V_1)$, thus

$$a_4=\|u\|_{L^4(0,T;V_1)}.$$

Thanks to (4.5)-(4.9) we estimate the quantities a_1, a_2, a_3 in Gronwall Lemma by

$$a_1=2c_4a_4,\ a_2=\frac{2r\|f\|^2}{\nu},\ a_3=\frac{r\|f\|^2}{\nu^2\lambda_1}+\frac{\|f\|^2}{\nu^3\lambda_1^2}.$$

Than we obtain

$$\|u_\varepsilon(t)\|_1^2 \leq (\frac{a_3}{r} + a_2)\exp(a_1) = R_1^2 \text{ for } t \geq t_0,\ t_0 \text{ as in (4.6).}$$

Hence, for any ball B_{R_1}, there exists a ball B_{δ_1}, in V_1 centered at origin with radius $R_1 > \delta_1 > \rho_1$ such that

$$S^\varepsilon(t)B_{R_1} \subset B_{\delta_1} \text{ for } t \geq t_1(B_{R_0}) = t_0(B_{R_0}) + 1 + \frac{1}{\nu\lambda_1}\log\frac{R_1^2 - \rho_1^2}{\delta_1^2 - \rho_1^2}.$$

The ball B_{δ_1} is said to be absorbing and invariant for the semigroup $S^\varepsilon(t)$.

Furthermore, if B is any bounded set of V_0, then $S^\varepsilon(t)B \subset B_{\delta_1}$ for $t \geq t_1(B, R_0)$, this shows the existence of an absorbing set in V_1. Since the embedding of V_1 in V_0 is compact, we deduce that $S^\varepsilon(t)$ maps a bounded set in V_0 into a compact set in V_0. In addition, the operators $S^\varepsilon(t)$ are uniformly compact for $t \geq t_1(B, R_0)$. That is,

$$\bigcup_{t\geq t_1} S^\varepsilon(t, 0, B_{R_0})$$

is relatively compact in V_0.

Due to a the standard procedure (cf., for example, [20, Theorem I.1.1] for details), one can prove that there is a global attractor $\mathcal{A}_\varepsilon$ for the operators $S^\varepsilon(t)$ for $\varepsilon \geq 0$,

Note that the global attractor $\mathcal{A}_\varepsilon$ must be contained in the absorbing balls V_0 and V_1

$$\mathcal{A}_\varepsilon = \bigcap_{t_1\geq 0}\overline{\bigcup_{t\geq t_1} B_{\delta_1}(t)} \subset B_{\delta_0} \cap B_{\delta_1}. \tag{4.13}$$

Theorem 4.4. *For fixed $\varepsilon \geq 0$, $u^\varepsilon \in B_{R_1} = \{u^\varepsilon(0) \in V_1; \|u^\varepsilon\|_1 \leq R_1\}$ and $f \in L^\infty C$ a time independent functions, π^ε is a continuous family of semiflows on X^0.*

Proof. Let convergent sequences ε_n, φ^n and f^n, with limits $\varepsilon_n \to \varepsilon_0$, (especially with $\varepsilon_0 = 0$), $\varphi^n \to \varphi_0$ in the X^0-topology and $f^n \to f^0$ in the $L^\infty C$-topology as $n \to \infty$, then

$$S^{\varepsilon_n}(f^n, t)\varphi^n \to S^{\varepsilon_0}\left(f^0, t\right)\varphi^0. \tag{4.14}$$

Let

$$S^{\varepsilon_n}(f^n, t)\varphi^n - S^{\varepsilon_0}\left(f^0, t\right)\varphi^0 = u^{\varepsilon_n}(t) - u^{\varepsilon_0}(t), \tag{4.15}$$

we obtain for $w_n = u^{\varepsilon_n}(t) - u^{\varepsilon_0}(t)$ and $g_n = f^n - f^0$

$$\partial_t w_n + \varepsilon_n A^l w_n + A w_n + B(u^{\varepsilon_n}, u^{\varepsilon_n}) - B(u^{\varepsilon_0}, u^{\varepsilon_0}) = g_n. \tag{4.16}$$

By taking inner product with w_n for above equation we get

$$\frac{1}{2}\frac{d}{dt}\|w_n\|^2 + \varepsilon_n\|A^{\frac{l}{2}}w_n\|^2 + \nu\|A^{\frac{1}{2}}w_n\|^2 = b(w_n, w_n, u^{\varepsilon_n}) + (g_n, w_n). \tag{4.17}$$

Using Young's inequality, we obtain

$$2(g_n, w_n) \leq \frac{2}{\nu}\|g_n\|^2 + \frac{\nu}{2}\|w_n\|_1^2,$$

By using inequalities (2.4) and Young's inequality we obtain

$$\begin{aligned}|2b(w_n, w_n, u^{\varepsilon_n})| &\leq 2c_1\|u^{\varepsilon_n}\|_1\|w_n\|_1^{\frac{3}{2}}\|w_n\|^{\frac{1}{2}} \\ &\leq \frac{c_1^4 R_1^4}{\nu^3}\|w_n\|^2 + \frac{3\nu}{4}\|w_n\|_1^2.\end{aligned}$$

Substituting the above result into (4.17), we obtain

$$\frac{d}{dt}\|w_n\|^2 + 2\varepsilon\|A^{\frac{l}{2}}w_n\|^2 + \frac{3\nu}{4}\|w_n\|_1^2 \leq \frac{c_1^4 R_1^4}{\nu^3}\|w_n\|^2 + \frac{2}{\nu}\|g_n\|^2. \tag{4.18}$$

We drop the positive terms $2\varepsilon\|A^{\frac{l}{2}}w_n\|^2$ and $\frac{3\nu}{4}\left\|w_\varepsilon\right\|_1^2$ to obtain the following differential inequality

$$\frac{d}{dt}\left\|w_n\right\|^2 \leq \frac{c_1^4 R_1^4}{\nu^3}\left\|w_n\right\|^2 + \frac{2}{\nu}\left\|g_n\right\|^2. \tag{4.19}$$

Applying now Gronwall's inequality to (4.19), for $t \geq 0$ we have

$$\begin{aligned} \left\|w_n\left(t\right)\right\|^2 &\leq \left\|w_n\left(0\right)\right\|^2 \exp T(\tfrac{c_1^4 R_1^4}{\nu^3}) \\ &+\frac{2}{\nu}\textstyle\int_0^t \exp T(\tfrac{c_1^4 R_1^4}{\nu^3})\left\|g_n\left(h\right)\right\|^2 dh \end{aligned} \tag{4.20}$$

finally we find

$$\left\|w_n\left(t\right)\right\|^2 \leq C_1\left\|w_n\left(0\right)\right\|^2 + \frac{2TC_1}{\nu}\left\|g_n\right\|^2 \tag{4.21}$$

for all t in compact sets in $[0,\infty)$, $C_1 = \exp T(\frac{c_1^4 R_1^4}{\nu^3})$. Since $f^n \to f^0$ in the $L^\infty C$-topology and $\varphi^n \to \varphi_0$ in the X^0-topology this means that $\|g_n\| \to 0$ and $\|w_n\left(0\right)\| \to 0$ as $n \to \infty$, it follows from (4.21) that

$$\left\|S^{\varepsilon_n}\left(f^n,t\right)\varphi^n - S^{\varepsilon_0}\left(f^0,t\right)\varphi^0\right\| \leq C_1\left\|u_0^{\varepsilon_n} - u_0^{\varepsilon_0}\right\|^2 + \frac{2TC_1}{\nu}\left\|f^n - f\right\|^2 \to 0, \text{ as } n \to \infty.$$

It follows that π^ε is continuous semiflows on X^0. Hence π^ε approximates π^0 on B_{R_1} uniformly on $[0,T]$. □

Regarding the existence of the attractor $\mathfrak{A}_\varepsilon$ when $\varepsilon > 0$, we use especially the related papers of Chepyzhov and Vishik, such as [14] to show that the system (4.1) possesses a global attractor. For $\varepsilon > 0$, we consider the trajectory space $\mathcal{K}_\varepsilon$ of the modified Navier-Stokes equations (4.1). $\mathcal{K}_\varepsilon$ is the union of all weak solutions $u^\varepsilon \in X^0$ that satisfy (4.1), see [12, formula (6.163)]. Using the described scheme in [14], we construct the spaces $\mathcal{S}_b$

$$\mathcal{S}_b = \{v\left(.\right) \in L^\infty\left(0,T;V_0\right) \cap L_b^2(0,T;V_1), \partial_t v\left(.\right) \in L_b^2(0,T;D\left(A^l\right)')\}$$

with norm

$$\|v\|_{\mathcal{S}_b} = \|v\|_{L_b^2(0,T;V_1)} + \|v\|_{L^\infty(0,T;V_0)} + \|\partial_t v\|_{L_b^2(0,T;D(A^l)')}$$

where

$$\|v\|_{L_b^2(0,T;V_1)} = \sup_{t\geq 0}(\int_t^{t+1}\left\|v\left(s\right)\right\|_1^2 ds)^{\frac{1}{2}}, \|v\|_{L^\infty(0,T;V_0)} = \operatorname{ess\,sup}_{t\geq 0}\|v\|$$

and

$$\|\partial_t v\|_{L_b^2(0,T;D(A^l)')} = \sup_{t\geq 0}(\int_t^{t+1}\left\|v\left(s\right)\right\|_{D(A^l)'}^2 ds)^{\frac{1}{2}}.$$

We need a topology in the space $\mathcal{K}_\varepsilon$. We define on X^0 the following sequential topology which we denote Γ.

By definition, a sequence of functions $\{v_n\} \subseteq X^0$ converges to a function $v \in X^0$ in the topology Γ as $n \to \infty$ if, for any $T > 0$, $v_n \to v$ weakly in $L^2(0,T;V_1)$; $v_n \to v$ weak-$*$ in $L^\infty\left(0,T;V_0\right)$ and $v_n \to v$ strongly in $L^2(0,T;V_0)$, as $n \to \infty$.

We consider the topology Γ on $\mathcal{K}_\varepsilon$. It is easy prove that the space $\mathcal{K}_\varepsilon$ is closed in Γ. From the definition of $\mathcal{K}_\varepsilon$, it follows that $\pi^\varepsilon\mathcal{K}_\varepsilon \subset \mathcal{K}_\varepsilon$ for all $t \geq 0$.

Proposition 4.5. *If $u^{\varepsilon}(t)$ is a solution of (4.1), then the following inequalities hold for all $t>0$*

$$\left\|u^{\varepsilon}(t)\right\|^{2}\leq e^{-\nu\lambda_{1}t}\left\|u_{0}^{\varepsilon}\right\|^{2}+\frac{\left\|f\right\|^{2}}{\nu^{2}\lambda_{1}^{2}},\tag{4.22}$$

$$\int_{t}^{t+1}\left\|u^{\varepsilon}(s)\right\|^{2}ds\leq\frac{e^{-\nu\lambda_{1}t}}{\nu\lambda_{1}}\left\|u_{0}^{\varepsilon}\right\|^{2}+\frac{\left\|f\right\|^{2}}{\nu^{2}\lambda_{1}^{2}},\tag{4.23}$$

$$\nu\int_{t}^{t+1}\left\|u^{\varepsilon}(s)\right\|_{1}^{2}ds\leq\frac{e^{-\nu\lambda_{1}t}}{\nu\lambda_{1}}\left\|u_{0}^{\varepsilon}\right\|^{2}+\frac{\left\|f\right\|^{2}}{\nu^{2}\lambda_{1}^{2}}+\frac{\left\|f\right\|^{2}}{\nu\lambda_{1}}.\tag{4.24}$$

Proof. Taking the inner product of (4.1) by $u^{\varepsilon}\in V_{2}$, we obtain

$$\frac{d}{dt}\left\|u^{\varepsilon}\right\|^{2}+2\varepsilon\left\|A^{l}u^{\varepsilon}\right\|^{2}+2\nu\left\|\nabla u^{\varepsilon}\right\|^{2}=2\left(f,u^{\varepsilon}\right).\tag{4.25}$$

Applying Young's inequality and using the Poincaré Lemma, we obtain

$$\frac{d}{dt}\left\|u^{\varepsilon}\right\|^{2}+\nu\left\|\nabla u^{\varepsilon}\right\|^{2}\leq\frac{\left\|f\right\|^{2}}{\nu\lambda_{1}}.\tag{4.26}$$

Using the Gronwall's inequality over $[0,\ t]$, we obtain (4.22). Integrating (4.22) over $[t,t+1]$ we find (4.23). Integrating (4.26) over $[t,t+1]$ we find

$$\nu\int_{t}^{t+1}\left\|\nabla u^{\varepsilon}(s)\right\|^{2}ds\leq\frac{\left\|f\right\|^{2}}{\nu\lambda_{1}}+\left\|u^{\varepsilon}(t)\right\|^{2}.$$

Applying inequality (4.22), we have (4.24). □

A simple consequence of [23, Lemma 3.6] is the following Lemma

Proposition 4.6. *Let $f\in V_{0}$. Then any solution $u^{\varepsilon}(t)$ of (4.1) satisfies*

$$\int_{t}^{t+1}\left\|\partial_{t}u^{\varepsilon}(s)\right\|_{D(A^{l})'}^{2}ds\leq C_{2},\tag{4.27}$$

C_{2} is a positive constant independent of ε .

Moreover, due to estimates (4.22) and (4.27), we also have the uniform estimate.

Proposition 4.7. *If $f\in V_{0}$, then any solution $u^{\varepsilon}(t)$ of problem (4.1) satisfies the inequality*

$$\left\|\pi^{\varepsilon}\left(u^{\varepsilon}\right)\right\|_{\mathcal{S}_{b}}^{2}\leq\frac{c_{7}e^{-\nu\lambda_{1}t}}{\nu\lambda_{1}}\left\|u^{\varepsilon}(0)\right\|^{2}+\frac{c_{7}\left\|f\right\|^{2}}{\nu^{2}\lambda_{1}^{2}}+C_{3}\tag{4.28}$$

where the positive constant C_{3} is independent of ε.

From Proposition 4.5 it follows that $\mathcal{K}_{\varepsilon}\subset\mathcal{S}_{b}$ for all $\varepsilon>0$ and for all $\tau>0$. Also Proposition 4.5 implies that the semigroup π^{ε} has absorbing set in $\mathcal{K}_{\varepsilon}$ for all $\varepsilon>0$ and for all $\tau>0$ (We note, that this absorbing set does not depend on ε, since the constant C_{3} in (4.28) is independent of ε), bounded in $\mathcal{S}_{b}$ and inequality (4.28) implies that absorbing set is compact in Γ. The continuity of π^{ε} is proved. These facts are sufficient to state that π^{ε} has a trajectory attractor $\mathfrak{A}_{\varepsilon}$. Such that $\mathfrak{A}_{\varepsilon}\subset\mathfrak{F}^{0}\times X^{0}$, bounded in $\mathcal{S}_{b}$ and compact in Γ. For a more detailed, see [14].

5. Upper semicontinuity of attractors

We now prove the robustness property for the trajectory attractor $\mathfrak{A}_\varepsilon$. We have shown in Theorem4.4 the continuity of the family of semiflows π^ε on X^0. Having done this, We can simply invoke Theorem 23.14 in [18] to complete the proof of the robustness for the family of attractors $\mathfrak{A}_\varepsilon$ at $\varepsilon = 0$. Clearly, it is sufficient to show that the small ε_0-neighbourhood of attractor $\mathfrak{A}_0$ is an absorbing set and that π^ε approximates π^0 on $B_{R_1} = \{u^\varepsilon(0) \in V_1; \|u_0^\varepsilon\|_1 \leq R_1\}$ uniformly on compact sets of $[0, \infty)$.

Theorem 5.1. *For $\varepsilon > 0$ the family of semiflows π^ε generated by the weak solutions of regularized 3D Navier-Stokes equations (1.1) admits a trajectory attractor $\{\mathfrak{A}_\varepsilon, 0 < \varepsilon \leq 1\}$ which attracts bounded sets of V_0 and is contained in the absorbing balls $B_{R_0} \cap B_{R_1}$ where R_0 and R_1 are independent of ε. Moreover, $d_{X^0}(\mathfrak{A}_\varepsilon, \mathfrak{A}_0) \to 0$, as $\varepsilon \to 0$.*

Proof. Let $N_{\varepsilon_0}(\mathfrak{A}_0)$ be the ε_0-neighborhood of $\mathfrak{A}_0$. Since $\mathfrak{A}_0$ is a attractor, for any bounded set $B_{R_0} = \{u(0) \in V_0; \|u(0)\| \leq R_0\} \subset V_0$, we have

$$d_{X^0}\left(\pi^0 B_{R_0}, \mathfrak{A}_0\right) \to 0, \text{ as } t \to \infty. \tag{5.1}$$

Thus, there exists $\varepsilon_0 > 0$ and $t > t_{\varepsilon_0}$ such that

$$d_{X^0}\left(\pi^0 B_{R_0}, \mathfrak{A}_0\right) \leq \frac{\varepsilon_0}{2}, \text{ for } t \geq t_{\varepsilon_0}. \tag{5.2}$$

Consequently

$$\pi^0(t) B_{R_0} \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } t \geq t_{\varepsilon_0}. \tag{5.3}$$

This shows that $N_{\varepsilon_0}(\mathfrak{A}_0)$ is an absorbing set. To establish the second step. Section. 3 implies that any ball $B_{R_1} = \{u_0^\varepsilon \in V_1; \|A^{\frac{1}{2}} u_0^\varepsilon(0)\| \leq R_1\}$ in V_1 with radius $R_1 > \rho_1$ will satisfy

$$\pi^\varepsilon(t) B_{R_1} \subset B_{R_1}, \text{ for } t \geq 0. \tag{5.4}$$

This means if $u_0^\varepsilon \in B_{R_1}$, then $\pi^\varepsilon(t) u_0^\varepsilon$ is defined and belongs to B_{R_1} for $t \geq 0$. The ball B_{R_1} is therefore invariant under the map π^ε. Since π^ε approximates π^0 on B_{R_1} uniformly on $[0, T]$, we have for any $\varepsilon_0 > 0$, there are $\varepsilon_1 > 0$ and $\tau_0 > 0$ such that

$$\pi^\varepsilon(B_{R_0} \cap B_{R_1}) \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1,\ t \geq \tau_0. \tag{5.5}$$

Since the attractor $\mathfrak{A}_\varepsilon$ is contained in $B_{R_0} \cap B_{R_1}$, an open neighborhood in the X^0 Fréchet space [18, Item (2) Theorem 23.14], we have

$$\pi^\varepsilon(\mathfrak{A}_\varepsilon) \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1,\ t \geq \tau_0. \tag{5.6}$$

Since $\mathfrak{A}_\varepsilon$ is an invariant set, we deduce that

$$\mathfrak{A}_\varepsilon \subset N_{\varepsilon_0}(\mathfrak{A}_0), \text{ for } 0 < \varepsilon < \varepsilon_1,\ t \geq \tau_0. \tag{5.7}$$

Moreover, since ε_0 is arbitrary, we obtain the upper semicontinuity of $\mathfrak{A}_\varepsilon$, at $\varepsilon = 0$

$$d_{X^0}(\mathfrak{A}_\varepsilon, \mathfrak{A}_0) \to 0, \text{ as } \varepsilon \to 0. \tag{5.8}$$

□

One can modify the argument described above so that the final result will have broader applicability by allowing the family of forcing functions to vary with ε,

for $\varepsilon > 0$. Thus, we consider the regularized Navier-Stokes system (1.1) with a perturbed external force f^ε in place of f, for $\varepsilon > 0$. Then (4.1) becomes

$$\begin{aligned} \partial_t u^\varepsilon + \varepsilon A^l u^\varepsilon + \nu A u^\varepsilon + B\left(u^\varepsilon, u^\varepsilon\right) = f^\varepsilon\left(x\right), &\quad \text{in} \quad \Omega \times (0,\infty) \\ u^\varepsilon\left(x\right) = u_0^\varepsilon, &\quad \text{in} \quad \Omega. \end{aligned} \tag{5.9}$$

We show that the trajectory attractor of the perturbed system (5.9) coincides with the trajectory attractor $\mathfrak{A}_\varepsilon$ of the unperturbed system (1.1). Our results rely on the work of Hale ([14]) who show that the limit behaviour is valid even through $\mathfrak{F}^\varepsilon$, where $\mathfrak{F}^\varepsilon$ denote a compact, translation invariant set of perturbed forcing functions to vary with ε, for $\varepsilon > 0$ and satisfy the condition

$$\omega(\mathcal{H}^+\left(f^\varepsilon\right)) = \omega(\mathcal{H}^+\left(f\right)). \tag{5.10}$$

Thus we would use $\mathfrak{F}^\varepsilon$ in place of $\mathfrak{F}^0$, for $\varepsilon > 0$. Moreover, by using a metric d on the $L^\infty C$-toplogy, see [18] for some samples, we can note that (5.10) is equivalent to saying that for every $\delta > 0$ there is an $\varepsilon_1 > 0$ and $T_\delta = T\left(\delta\right) \geq 0$ such that

$$d_{X^0}(f^\varepsilon, \mathfrak{F}^0) \leq \delta, \text{ for } 0 < \varepsilon \leq \varepsilon_1 \text{ and } f^\varepsilon \in \mathfrak{F}^\varepsilon$$

for any $t \geq T_\delta$, that is

$$\mathfrak{F}^\varepsilon \subset N_\delta\left(\mathfrak{F}^0\right), \text{ for } 0 < \varepsilon \leq \varepsilon_1, \tag{5.11}$$

where N_δ denotes the δ-neighborhood of $\mathfrak{F}^0$ in $L^\infty C$. The resulting argument for robustness will then depend on two parameters $\lambda = (\varepsilon, \delta)$, where $\lambda \to (0,0)$.

The following statement generalizes Theorem 5.1

Theorem 5.2. *Under the above conditions, the trajectory attractor of the perturbed 3D Navier-Stokes system (5.9) coincides with the trajectory attractor $\mathfrak{A}_\varepsilon$ of the non-perturbed system (4.1). Moreover, the perturbed attractor of (5.9) is upper semicontinuous with respect to ε at $\varepsilon = 0$.*

Proof. The existence of trajectory attractor $\mathfrak{A}_\varepsilon$ is treated above. The proof follows from formulas (5.10), (5.11) and Theorem 5.1 . □

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Department of Mathematics and Computer Science , University of Djelfa , Algeria.
E-mail address: `younsihafid@gmail.com`